The hamburger theorem

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In memory of Ferran Hurtado and Jiří Matoušek

Abstract

We generalize the ham sandwich theorem to $d+1$ measures on $\mathbb{R}^d$ as follows. Let $\mu_1, \mu_2, \dots, \mu_{d+1}$ be absolutely continuous finite Borel measures on $\mathbb{R}^d$. Let $\omega_i = \mu_i(\mathbb{R}^d)$ for $i \in [d+1]$, $\omega = \min\{\omega_i; i \in [d+1]\}$ and assume that $\sum_{j=1}^{d+1} \omega_j = 1$. Assume that $\omega_i \le 1/d$ for every $i \in [d+1]$. Then there exists a hyperplane h such that each open halfspace H defined by h satisfies $\mu_i(H) \le (\sum_{j=1}^{d+1} \mu_j(H))/d$ for every $i \in [d+1]$ and $\sum_{j=1}^{d+1} \mu_j(H) \ge \min\{1/2, 1-d\omega\} \ge 1/(d+1)$. As a consequence we obtain that every $(d+1)$-colored set of nd points in $\mathbb{R}^d$ such that no color is used for more than n points can be partitioned into n disjoint rainbow $(d-1)$-dimensional simplices.

Keywords: Borsuk–Ulam theorem; ham sandwich theorem; hamburger theorem; absolutely continuous Borel measure; colored point set.

1 Introduction

It is well-known that if n red points and n blue points are given in the plane in general position, then there exists a noncrossing perfect matching on these points where each edge is a straight-line segment and connects a red point with a blue point. Akiyama and Alon [2] generalized this result to higher dimensions as follows.

For a positive integer m, we write $\mathbb{R}^m$ for the m-dimensional Euclidean space and $[m]$ for the set $\{1, 2, \dots, m\}$.

Theorem 1 (Akiyama and Alon [2]). *Let $d \ge 2$ and $n \ge 2$ be integers, and for each $i \in [d]$, let X_i be a set of n points in $\mathbb{R}^d$ such that all X_i are pairwise disjoint and no $d+1$ points of $X_1 \cup X_2 \cup \dots \cup X_d$ are contained in a hyperplane. Then there exist n pairwise disjoint $(d-1)$-dimensional simplices, each of which contains precisely one vertex from each X_i, $i \in [d]$.*

The planar version of Theorem 1 follows, for example, from the simple fact that a shortest geometric red-blue perfect matching is noncrossing [2]. This elementary metric argument does

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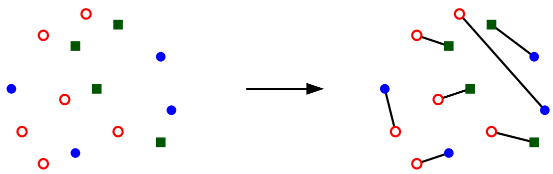

Figure 1: A noncrossing geometric properly colored perfect matching.

not generalize to higher dimensions, however. Akiyama and Alon [2] proved Theorem 1 using the ham sandwich theorem; see Subsection 1.1.

Aichholzer et al. [1] and Kano, Suzuki and Uno [13] extended the planar version of Theorem 1 to an arbitrary number of colors as follows.

Theorem 2 (Aichholzer et al. [1]; Kano, Suzuki and Uno [13]). *Let $r \geq 3$ and $n \geq 1$ be integers. Let $X_1, X_2, \ldots, X_r$ be r disjoint point sets in the plane. Assume that no three points of $X_1 \cup X_2 \cup \cdots \cup X_r$ lie on a line, $\sum_{i=1}^{r} |X_i| = 2n$, and $|X_i| \leq n$ for every $i \in [r]$. Then there exists a noncrossing geometric perfect matching on $X_1 \cup X_2 \cup \cdots \cup X_r$ where every edge connects two points from different sets X_i and X_j.*

Aichholzer et al. [1] proved Theorem 2 by the same metric argument as the in case of two colors. Kano, Suzuki and Uno [13] first proved Theorem 2 for three colors, by induction using a result on partitions of 3-colored point sets on a line. Then they derived the case of four or more colors by merging the smallest color classes together.

Kano and Suzuki [12] made the following conjecture generalizing Theorem 1 and Theorem 2.

Conjecture 3 (Kano and Suzuki [12]). *Let $r \geq d \geq 3$ and $n \geq 1$ be integers. Let $X_1, X_2, \ldots, X_r$ be r disjoint point sets in $\mathbb{R}^d$. Assume that no $d+1$ points of $X_1 \cup X_2 \cup \cdots \cup X_r$ lie in a hyperplane, $\sum_{i=1}^{r} |X_i| = dn$, and $|X_i| \leq n$ for every $i \in [r]$. Then there exist n pairwise disjoint $(d-1)$-dimensional simplices, each of them having d vertices in d distinct sets X_i.*

Conjecture 3 holds when $r = d$ by Theorem 1 or $d = 2$ by Theorem 2.

In this paper we prove Conjecture 3 for every $d \geq 2$ and $r = d + 1$. We restate it as the following theorem.

Theorem 4. *Let $d \geq 2$ and $n \geq 1$ be integers. Let $X_1, X_2, \ldots, X_{d+1}$ be $d+1$ disjoint point sets in $\mathbb{R}^d$. Assume that no $d+1$ points of $X_1 \cup X_2 \cup \cdots \cup X_{d+1}$ lie in a hyperplane, $\sum_{i=1}^{d+1} |X_i| = dn$, and $|X_i| \leq n$ for every $i \in [d+1]$. Then there exist n pairwise disjoint $(d-1)$-dimensional simplices, each of them having d vertices in d distinct sets X_i.*

The proof of Theorem 4 (see Section 3) provides yet another different proof of Theorem 2.

Many related results and problems on colored point sets can be found in a survey by Kaneko and Kano [11].

1.1 Simultaneous partitions of measures

We denote by S^n the n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$, that is, $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1}; \|\mathbf{x}\| = 1\}$.

The Borsuk–Ulam theorem plays an important role throughout this paper.

Theorem 5 (The Borsuk–Ulam theorem [14, Theorem 2.1.1]). *Let $f : S^n \to \mathbb{R}^n$ be a continuous mapping. If $f(-\mathbf{u}) = -f(\mathbf{u})$ for all $\mathbf{u} \in S^n$, then there exists a point $\mathbf{v} \in S^n$ such that $f(\mathbf{v}) = \mathbf{0} = (0, 0, \dots, 0)$.*

Informally speaking, the ham sandwich theorem states that a sandwich made of bread, ham and cheese can be cut by a single plane, bisecting the mass of each of the three ingredients exactly in half. According to Beyer and Zardecki [3], the ham sandwich theorem was conjectured by Steinhaus and appeared as Problem 123 in The Scottish Book [15]. Banach gave an elementary proof of the theorem using the Borsuk–Ulam theorem for S^2. A more direct proof can be obtained from the Borsuk–Ulam theorem for S^3 [14]. Stone and Tukey [16] generalized the ham sandwich theorem to d-dimensional sandwiches made of d ingredients as follows.

Theorem 6 (The ham sandwich theorem [16], [14, Theorem 3.1.1]). *Let $\mu_1, \mu_2, \dots, \mu_d$ be d absolutely continuous finite Borel measures on $\mathbb{R}^d$. Then there exists a hyperplane h such that each open halfspace H defined by h satisfies $\mu_i(H) = \mu_i(\mathbb{R}^d)/2$ for every $i \in [d]$.*

Stone and Tukey [16] proved more general versions of the ham sandwich theorem, including a version for Carathéodory outer measures and more general cutting surfaces. Cox and McKelvey [6] and Hill [9] generalized the ham sandwich theorem to general finite Borel measures, which include measures with finite support. For these more general measures, the condition $\mu_i(H) = \mu_i(\mathbb{R}^d)/2$ must be replaced by the inequality $\mu_i(H) \le \mu_i(\mathbb{R}^d)/2$. Breuer [5] gave sufficient conditions for the existence of more general splitting ratios. In particular, he showed that for absolutely continuous measures whose supports can be separated by hyperplanes, there is a hyperplane splitting the measures in any prescribed ratio. See Matoušek's book [14] for more generalizations of the ham sandwich theorem and other related partitioning results.

To prove Theorem 4, we follow the approach by Akiyama and Alon [2]. To this end, we need to generalize the ham sandwich theorem to $d+1$ measures on $\mathbb{R}^d$. Clearly, it is not always possible to bisect all $d+1$ measures by a single hyperplane, for example, if each measure is concentrated in a small ball around a different vertex of a regular simplex.

Let $r \ge d$ and let $\mu_1, \mu_2, \dots, \mu_r$ be finite Borel measures on $\mathbb{R}^d$. We say that $\mu_1, \mu_2, \dots, \mu_r$ are *balanced* in a subset $X \subseteq \mathbb{R}^d$ if for every $i \in [r]$, we have

$$\mu_i(X) \le \frac{1}{d} \cdot \sum_{j=1}^{r} \mu_j(X).$$

Theorem 7 (The hamburger theorem). *Let $d \ge 2$ be an integer. Let $\mu_1, \mu_2, \dots, \mu_{d+1}$ be absolutely continuous finite Borel measures on $\mathbb{R}^d$. Let $\omega_i = \mu_i(\mathbb{R}^d)$ for $i \in [d+1]$ and $\omega = \min\{\omega_i; i \in [d+1]\}$. Assume that $\sum_{j=1}^{d+1} \omega_j = 1$ and that $\mu_1, \mu_2, \dots, \mu_{d+1}$ are balanced in $\mathbb{R}^d$. Then there exists a hyperplane h such that for each open halfspace H defined by h, the measures $\mu_1, \mu_2, \dots, \mu_{d+1}$ are balanced in H and $\sum_{j=1}^{d+1} \mu_j(H) \ge \min\{1/2, 1-d\omega\} \ge 1/(d+1)$.*

Moreover, setting $t = \min\{1/(2d), 1/d - \omega\}$ and assuming that $\omega_{d+1} = \omega$, the vector $(\mu_1(H), \mu_2(H), \dots, \mu_{d+1}(H))$ is a convex combination of the vectors $(t, t, \dots, t, 0)$ and $(\omega_1 - t, \omega_2 - t, \dots, \omega_d - t, \omega_{d+1})$.

Our choice of the name for Theorem 7 is motivated by the fact that compared to a typical ham sandwich, a typical hamburger consists of more ingredients, such as bread, beef, bacon, and salad.

Note that the lower bound $\min\{1/2, 1-d\omega\}$ on the total measure of the two halfspaces is tight: consider, for example, $d+1$ measures such that each of them is concentrated in a small ball centered at a different vertex of the unit d-dimensional simplex.

2 Proof of the hamburger theorem

In this section we prove Theorem 7. We follow the proof of the ham sandwich theorem for measures from [14].

The set of open half-spaces in $\mathbb{R}^d$, together with the empty set and the whole space $\mathbb{R}^d$, has a natural topology of the sphere S^d. We use the following parametrization.

Let $\mathbf{u} = (u_0, u_1, \dots, u_d)$ be a point from the sphere S^d, that is, $u_0^2 + u_1^2 + \cdots + u_d^2 = 1$. If $|u_0| < 1$, then at least one of the coordinates $u_1, u_2, \dots, u_d$ is nonzero, and we define two halfspaces as follows:

$$H^-(\mathbf{u}) = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d; u_1x_1 + u_2x_2 + \cdots + u_dx_d < u_0\},$$
$$H^+(\mathbf{u}) = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d; u_1x_1 + u_2x_2 + \cdots + u_dx_d > u_0\}.$$

We also define a hyperplane $h(\mathbf{u})$ as the common boundary of $H^-(\mathbf{u})$ and $H^+(\mathbf{u})$. For the two remaining points $(1, 0, \dots, 0)$ and $(-1, 0, \dots, 0)$, we set

$$H^-(1,0,0,\dots,0) = \mathbb{R}^d, \qquad H^+(1,0,0,\dots,0) = \emptyset,$$
$$H^-(-1,0,0,\dots,0) = \emptyset, \qquad H^+(-1,0,0,\dots,0) = \mathbb{R}^d.$$

Note that antipodal points on S^d correspond to complementary half-spaces; that is, $H^-(\mathbf{u}) = H^+(-\mathbf{u})$ for every $\mathbf{u} \in S^d$.

We define a function $f = (f_1, f_2, \dots, f_{d+1}) : S^d \to \mathbb{R}^{d+1}$ by

$$f_i(\mathbf{u}) = \mu_i(H^-(\mathbf{u})).$$

Since the measures μ_i are absolutely continuous, $\mu_i(h(\mathbf{u})) = 0$ for every hyperplane $h(\mathbf{u})$. This implies that f is continuous [14].

The image of f lies in the box $B = \prod_{i=1}^{d+1}[0, \omega_i]$. Moreover, f maps antipodal points of the sphere to points symmetric about the center of B. The *target polytope* is the subset of points $(y_1, y_2, \dots, y_{d+1})$ of B satisfying the inequalities

$$y_i \le \frac{1}{d} \cdot \sum_{j=1}^{d+1} y_j \qquad \text{and} \qquad \omega_i - y_i \le \frac{1}{d} \cdot \sum_{j=1}^{d+1} (\omega_j - y_j). \tag{1}$$

See Figure 2, left. The subset of the target polytope satisfying the inequalities

$$\min\{1/2, 1-d\omega\} \le y_1 + y_2 + \cdots + y_{d+1} \le 1 - \min\{1/2, 1-d\omega\} \tag{2}$$

is called the *truncated target polytope*; see Figure 2, right. Our goal is to show that the image of f intersects the truncated target polytope.

We first show a proof using the notion of a *degree* of a map between spheres. Then we modify it so that it uses only the Borsuk–Ulam theorem.

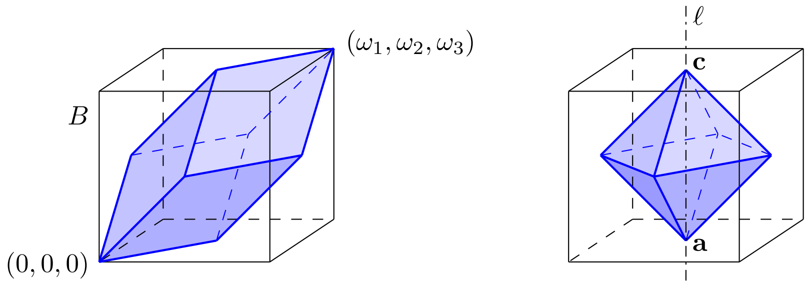

Figure 2: Left: the target polytope inside B for $d = 2$ and $\omega_1 = \omega_2 = \omega_3 = 1/3$. Right: the truncated target polytope corresponding to hyperplanes cutting at least $1/3$ of the total measure on both sides. The segment $\mathbf{ac}$, which is the intersection of the line ℓ with B, is contained in the truncated target polytope.

Let $\mathbf{b} = (\omega_1/2, \omega_2/2, \dots, \omega_{d+1}/2)$ be the center of B. The map $g = f - \mathbf{b}$ is antipodal, that is, $g(-\mathbf{u}) = -g(\mathbf{u})$ for every $\mathbf{u} \in S^d$. Clearly, $\mathbf{b}$ satisfies both (1) and (2) and thus it belongs to the truncated target polytope. Hence, if $\mathbf{0}$ is in the image of g, then any hyperplane $h(\mathbf{u})$ such that $g(\mathbf{u}) = \mathbf{0}$ satisfies the theorem.

For the rest of the proof we may assume that $\mathbf{0}$ is not in the image of g. Then we can define an antipodal map $\tilde{g} : S^d \to S^d$ as

$$\tilde{g}(\mathbf{u}) = \frac{g(\mathbf{u})}{\|g(\mathbf{u})\|}.$$

Using the fact that every antipodal map from S^d to itself has odd degree [8, Proposition 2B.6.], we conclude that $\tilde{g}$ is surjective. Hence, the image of g intersects every line passing through the origin, equivalently, the image of f intersects every line passing through $\mathbf{b}$. Therefore, it is sufficient to find a line ℓ through $\mathbf{b}$ such that $\ell \cap B$ belongs to the truncated target polytope.

Without loss of generality, we may assume that $\omega_1 \geq \omega_2 \geq \dots \geq \omega_{d+1} = \omega$. Let

$$t = \min\left\{\frac{1}{2d}, \frac{1}{d} - \omega_{d+1}\right\}.$$

We define ℓ as the line containing the points

$$\begin{aligned} \mathbf{a} &= (t, t, \dots, t, 0) \quad \text{and} \\ \mathbf{c} &= (\omega_1 - t, \omega_2 - t, \dots, \omega_d - t, \omega_{d+1}). \end{aligned}$$

Since the measures $\mu_1, \mu_2, \dots, \mu_{d+1}$ are balanced in $\mathbb{R}^d$, we have $\omega_i \leq 1/d$ for all $i \in [d+1]$. Thus $\omega_d + \omega_{d+1} = 1 - (\omega_1 + \omega_2 + \dots + \omega_{d-1}) \geq 1/d$ and consequently $\omega_d \geq 1/(2d) \geq t$. This implies that both points $\mathbf{a}$ and $\mathbf{c}$ lie in B. Moreover, $\mathbf{a}$ and $\mathbf{c}$ lie on the opposite facets of B and they are symmetric around the center $\mathbf{b}$. The points $\mathbf{a}$ and $\mathbf{c}$ both satisfy (1) since $(\omega_1 - t) + (\omega_2 - t) + \dots + (\omega_d - t) + \omega_{d+1} = 1 - dt \geq d\omega_{d+1}$. They also both satisfy (2) since $dt \leq 1 - dt$. Therefore, the segment $\mathbf{ac}$ is the intersection of the line ℓ with B and it is contained in the truncated target polytope.

Now we show how to replace the degree argument with the application of the Borsuk–Ulam theorem. Let π_ℓ be a projection of $\mathbb{R}^{d+1}$ in the direction of the line ℓ to a d-dimensional

subspace orthogonal to ℓ, which we identify with $\mathbb{R}^d$. Define a map $g' : S^d \to \mathbb{R}^d$ by $g'(\mathbf{u}) = \pi_\ell(g(\mathbf{u}))$. The map g' is antipodal, and so by the Borsuk–Ulam theorem, there exists $\mathbf{u} \in S^d$ such that $g'(\mathbf{u}) = \mathbf{0}$, which means that $f(\mathbf{u}) \in \ell$. This concludes the proof.

3 A discrete version of the hamburger theorem

Theorem 4 follows by induction from the following discrete analogue of the hamburger theorem.

We say that point sets $X_1, X_2, \ldots, X_r$ are *balanced* in a subset $S \subseteq \mathbb{R}^d$ if for every $i \in [r]$, we have

$$|S \cap X_i| \leq \frac{1}{d} \cdot \sum_{j=1}^{r} |S \cap X_j|.$$

Theorem 8. *Let $d \geq 2$ and $n \geq 2$ be integers. Let $X_1, X_2, \ldots, X_{d+1} \subset \mathbb{R}^d$ be $d+1$ disjoint point sets balanced in $\mathbb{R}^d$. Assume that no $d+1$ points of $X_1 \cup X_2 \cup \cdots \cup X_{d+1}$ lie in a hyperplane and that $\sum_{i=1}^{d+1} |X_i| = dn$. Then there exists a hyperplane h disjoint with each X_i such that for each open halfspace H determined by h, the sets $X_1, X_2, \ldots, X_{d+1}$ are balanced in H and $\sum_{i=1}^{d+1} |H \cap X_i|$ is a positive integer multiple of d.*

While preparing the final version of this paper, we have learned that Biniaz, Maheshwari, Nandy and Smid [4] independently proved the plane version of Theorem 8, and they gave a linear-time algorithm for computing the cutting line.

3.1 Proof of Theorem 8.

Let $X = \bigcup_{i=1}^{d+1} X_i$. Replace each point $\mathbf{p} \in X$ by an open ball $B(\mathbf{p})$ of a sufficiently small radius $\delta > 0$ centered in $\mathbf{p}$, so that no hyperplane intersects more than d of these balls. We will apply the hamburger theorem for suitably defined measures supported by the balls $B(\mathbf{p})$. Rather than taking the same measure for each of the balls, we use a variation of the trick used by Elton and Hill [7]. For each $\mathbf{p} \in X$ and $k \geq 1$, we choose a number $\beta_k(\mathbf{p}) \in (1-1/k, 1+1/k)$ so that the following two conditions are satisfied.

I) For every $i \in [d+1]$, we have $\sum_{\mathbf{p} \in X_i} \beta_k(\mathbf{p}) = |X_i|$.

II) Let $\omega_i = |X_i|/|X|$ for $i \in [d+1]$, suppose that $\omega_{d+1} = \min\{\omega_i; i \in [d+1]\}$ and let $t = \min\{1/(2d), 1/d - \omega_{d+1}\}$. For every $i, j \in [d+1]$, $i \neq j$, and for every pair of proper nonempty subsets $Y \subset X_i$ and $Z \subset X_j$, there is no vector $(\gamma_1, \gamma_2, \ldots, \gamma_{d+1}) \in \mathbb{R}^{d+1}$ that is a convex combination of the vectors $(t, t, \ldots, t, 0)$ and $(\omega_1 - t, \omega_2 - t, \ldots, \omega_d - t, \omega_{d+1})$ and satisfies $\gamma_i = \sum_{\mathbf{p} \in Y} \beta_k(\mathbf{p})$ and $\gamma_j = \sum_{\mathbf{p} \in Z} \beta_k(\mathbf{p})$.

Now let $k \geq 1$ be a fixed integer. For each $i \in [d+1]$, let $\mu_{i,k}$ be the measure supported by the closure of $\bigcup_{\mathbf{p} \in X_i} B(\mathbf{p})$ such that it is uniform (that is, equal to a multiple of the Lebesgue measure) on each of the balls $B(\mathbf{p})$ and $\mu_{i,k}(B(\mathbf{p})) = \beta_k(\mathbf{p})$.

By the condition I), the measures $\mu_{1,k}, \mu_{2,k}, \ldots, \mu_{d+1,k}$ are balanced in $\mathbb{R}^d$. We may thus apply the hamburger theorem to the normalized collection of measures $\mu_{i,k}/|X|$. Let h_k be the resulting hyperplane. We distinguish two cases.

1) We have $\mu_{i,k}(H) = 0$ for some $i \in [d+1]$ and some halfspace H determined by h_k. Since the measures $\mu_{1,k}, \mu_{2,k}, \ldots, \mu_{d+1,k}$ are balanced in H, there is an $\alpha > 0$ such that $\mu_{j,k}(H) = \alpha$ for every $j \in [d+1] \setminus \{i\}$.

2) The hyperplane h_k splits each measure $\mu_{i,k}$ in a nontrivial way. By the condition II), h_k intersects the support of exactly d of the measures $\mu_{i,k}$.

For each $i \in [d+1]$, let μ_i be the limit of the measures $\mu_{i,k}$ when k grows to infinity; that is, μ_i is uniform on every ball $B(\mathbf{p})$ such that $\mathbf{p} \in X_i$ and $\mu_i(B(\mathbf{p})) = 1$. Since the supports of all the measures $\mu_{i,k}$ are uniformly bounded, there is an increasing sequence $\{k_m; m = 1, 2, \dots\}$ such that the sequence of hyperplanes h_{k_m} has a limit h'. More precisely, if $h_{k_m} = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v}_m = c_m\}$ where $\mathbf{v}_m \in S^{d-1}$, then $h' = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v} = c\}$ where $\mathbf{v} = \lim_{m\to\infty} \mathbf{v}_m$ and $c = \lim_{m\to\infty} c_m$. By the absolute continuity of the measures, the measures $\mu_1, \mu_2, \dots, \mu_{d+1}$ are balanced in each of the two halfspaces determined by h', and the total measure $\sum_{j=1}^{d+1} \mu_j$ of each of the two halfspaces is at least $n/(d+1)$. One of the cases 1) or 2) occurred for infinitely many hyperplanes h_{k_m}, $m \geq 1$. We distinguish the two possibilities.

Suppose that case 1) occurred infinitely many times. Then there is an $i \in [d+1]$ such that $\mu_{i,k_m}(H^+_{k_m}) = 0$ occurred for infinitely many $m \geq 1$ or $\mu_{i,k_m}(H^-_{k_m}) = 0$ occurred for infinitely many $m \geq 1$, where $H^+_{k_m} = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v}_m > c_m\}$ and $H^-_{k_m} = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v}_m < c_m\}$. By the absolute continuity of the measures, there is an $\alpha \geq n/(d+1) > 0$ such that for one of the halfspaces H determined by h' and for every $j \in [d+1] \setminus \{i\}$, we have $\mu_j(H) = \alpha$.

If h' is disjoint from all the balls $B(\mathbf{p})$, $\mathbf{p} \in X$, then this hyperplane satisfies the conditions of the theorem. Otherwise, h' intersects one ball from the support of each of the measures μ_j, $j \in [d+1] \setminus \{i\}$. Let $\tilde{h}$ be the translation of h' that touches each of these d balls and such that $\tilde{h}$ is in the complement of H. Let $\tilde{H}$ be the open halfspace determined by $\tilde{h}$ containing H, and let $\tilde{H}'$ be the open halfspace opposite to $\tilde{H}$. Then for every $j \in [d+1] \setminus \{i\}$, we have $\mu_j(\tilde{H}) = \lceil \alpha \rceil$. In particular, the sets $X_1, X_2, \dots, X_{d+1}$ are balanced in $\tilde{H}$ and $0 < |\tilde{H} \cap X| = d\lceil \alpha \rceil < dn$. It remains to show that $X_1, X_2, \dots, X_{d+1}$ are also balanced in $\tilde{H}'$. Let H' be the open halfspace opposite to H. Since the measures $\mu_1, \mu_2, \dots, \mu_{d+1}$ are balanced in H', we have

$$\begin{aligned}
\mu_i(\tilde{H}') = \mu_i(H') \leq \frac{1}{d} \cdot \sum_{j=1}^{d+1} \mu_j(H') &= \frac{1}{d} \cdot \left(\sum_{j=1}^{d+1} \mu_j(\tilde{H}') \right) + (\lceil \alpha \rceil - \alpha) \\
&= (n - \lceil \alpha \rceil) + (\lceil \alpha \rceil - \alpha) = n - \alpha.
\end{aligned}$$

Since $\mu_i(\tilde{H}') = |X_i|$, we can replace the upper bound by the nearest integer, and thus we have

$$\mu_i(\tilde{H}') \leq n - \lceil \alpha \rceil = \frac{1}{d} \cdot \left(\sum_{j=1}^{d+1} \mu_j(\tilde{H}') \right).$$

Finally, suppose that case 2) occurred infinitely many times. There is an $i \in [d+1]$ such that for infinitely many $m \geq 1$, the hyperplane h_{k_m} intersects the support of each measure μ_{j,k_m}, $j \in [d+1] \setminus \{i\}$. In particular, for each $j \in [d+1] \setminus \{i\}$, there is a point $\mathbf{p}_j \in X_j$ such that for infinitely many $m \geq 1$, the hyperplane h_{k_m} intersects each of the balls $B(\mathbf{p}_j)$. It follows that h' intersects or touches each of the balls $B(\mathbf{p}_j)$.

Call a hyperplane in $\mathbb{R}^d$ *admissible* if it intersects or touches each of the balls $B(\mathbf{p}_j)$, $j \in [d+1] \setminus \{i\}$, and the measures $\mu_1, \mu_2, \dots, \mu_{d+1}$ are balanced in each of the two open halfspaces determined by the hyperplane. Among all admissible hyperplanes, let h be a hyperplane that intersects the minimum possible number of the balls $B(\mathbf{p}_j)$; equivalently, h touches as many of the balls $B(\mathbf{p}_j)$ as possible.

Let H and H' be the two open halfspaces determined by h. For $j \in [d+1]$, call a measure μ_j *saturated in* a set $S \subseteq \mathbb{R}^d$ if $\mu_j(S) = (\sum_{k=1}^{d+1} \mu_k(S))/d$. Call a measure μ_j *saturated* if μ_j is saturated in H or in H'. We may assume, under the conditions above, that h is chosen so that the number of saturated measures is the maximum possible.

Observation 9. *If μ_j and $\mu_{j'}$ are saturated and h intersects $B(\mathbf{p}_j)$, then $\mu_j(H \cap B(\mathbf{p}_j)) = \mu_{j'}(H \cap B(\mathbf{p}_{j'}))$.*

Proof. If μ_j and $\mu_{j'}$ are saturated in the same halfspace, say, H, then $\mu_j(H) = \mu_{j'}(H)$. If μ_j is saturated in H and $\mu_{j'}$ is saturated in H', then $\mu_j(H) + \mu_{j'}(H') = n$. The observation follows since for each of the measures, each of the balls $B(\mathbf{p})$, $\mathbf{p} \in X$, has measure 0 or 1, and h intersects at most one ball from the support of each of the measures. $\square$

Observation 10. *There is at most one measure μ_j such that μ_j is not saturated and h intersects $B(\mathbf{p}_j)$.*

Proof. Suppose that there are $j, j' \in [d+1] \setminus \{i\}$, $j \neq j'$, such that neither of μ_j, $\mu_{j'}$ is saturated and h intersects both $B(\mathbf{p}_j)$ and $B(\mathbf{p}_{j'})$. We can rotate the hyperplane h while keeping the values $\mu_k(H \cap B(\mathbf{p}_k))$ constant for each $k \in [d+1] \setminus \{i, j, j'\}$, keeping the value $\mu_j(H \cap B(\mathbf{p}_j)) + \mu_{j'}(H \cap B(\mathbf{p}_{j'}))$ constant, and decreasing the value $\mu_j(H \cap B(\mathbf{p}_j))$ while increasing the value $\mu_{j'}(H \cap B(\mathbf{p}_{j'}))$, until one of the measures μ_j, $\mu_{j'}$ becomes saturated or h does not intersect $B(\mathbf{p}_j)$ or $B(\mathbf{p}_{j'})$ anymore. Observe that h is admissible all the time during the rotation. $\square$

Observation 11. *If μ_i is saturated, or if, for some $j \in [d+1] \setminus \{i\}$, μ_j is saturated and h does not intersect $B(\mathbf{p}_j)$, then h touches each of the balls $B(\mathbf{p}_k)$, $k \in [d+1] \setminus \{i\}$. Moreover, $|H \cap X|$ is an integer multiple of d.*

Proof. The observation follows by Observations 9 and 10 and from the fact that $\sum_{k=1}^{d+1} \mu_k(S) = d\mu_j(S)$ for every measure μ_j saturated in S. $\square$

Observation 12. *If μ_j is saturated and $j \neq i$, then h does not intersect $B(\mathbf{p}_j)$.*

Proof. Suppose for contrary that μ_j is saturated and h intersects $B(\mathbf{p}_j)$. If all the measures μ_k, $k \in [d+1] \setminus \{i\}$, are saturated, then by Observation 9, h intersects each of the balls $B(\mathbf{p}_k)$, $k \in [d+1] \setminus \{i\}$, and moreover, there is an $\alpha \in (0,1)$ such that $\mu_k(H \cap B(\mathbf{p}_k)) = \alpha$, for each $k \in [d+1] \setminus \{i\}$. In this case, we can move h to a hyperplane $\tilde{h}$ that touches each of the balls $B(\mathbf{p}_k)$. The measures μ_k, $k \in [d+1] \setminus \{i\}$ are saturated all the time during the translation, and μ_i can become saturated only at the end when $h = \tilde{h}$ by Observation 11.

Thus, at most $d-1$ of the measures μ_k, $k \in [d+1] \setminus \{i\}$, are saturated. Let $\sigma \subset [d+1] \setminus \{i\}$ be the set of all $k \in [d+1] \setminus \{i\}$ such that μ_k is saturated. Let $\alpha = \mu_j(H \cap B(\mathbf{p}_j))$. Let $\beta = \sum_{k \in [d+1] \setminus (\{i\} \cup \sigma)} \mu_k(H \cup B(\mathbf{p}_k))$ and $\beta' = \sum_{k \in [d+1] \setminus (\{i\} \cup \sigma)} \mu_k(H' \cup B(\mathbf{p}_k))$. Since $\beta + \beta' = d - |\sigma|$, we have $\beta \geq \alpha(d - |\sigma|)$ or $\beta' \geq (1-\alpha)(d - |\sigma|)$. In the first case, we rotate h while keeping the values $\mu_k(H \cap B(\mathbf{p}_k))$ equal for all $k \in \sigma$, keeping all the measures μ_k, $k \in \sigma$, balanced, keeping the condition of Observation 10, and decreasing $\mu_j(H \cap B(\mathbf{p}_j))$ towards 0. In the other case, we keep the same conditions except that we increase $\mu_j(H \cap B(\mathbf{p}_j))$ towards 1. The hyperplane h stays admissible all the time during the rotation, but eventually it touches more of the balls $B(\mathbf{p}_k)$ than in the beginning, or some measure μ_k, $k \in [d+1] \setminus \sigma$, becomes saturated. $\square$

Observation 13. *The hyperplane h touches each of the balls $B(\mathbf{p}_j)$, $j \in [d+1] \setminus \{i\}$.*

Proof. By Observations 10 and 12, h intersects at most one of the balls, say, $B(\mathbf{p}_j)$, and in that case μ_j is not saturated. By Observation 11, none of the measures μ_k, $k \in [d+1]$, is saturated either. We may thus rotate h while keeping the property that h touches each of the balls $B(\mathbf{p}_k)$, $k \in [d+1] \setminus \{i, j\}$, until h touches $B(\mathbf{p}_j)$, too. The hyperplane stays admissible all the time during the rotation. □

If h satisfies the property that $|H \cap X|$ is divisible by d, then we are finished, since both $H \cap X_i$ and $H' \cap X_i$ are nonempty. Otherwise, by Observation 11, none of the measures μ_j, $j \in [d+1]$, is saturated. Therefore, we can rotate h in order to move some of the balls $B(\mathbf{p}_j)$ from one side of h to the other, while keeping h admissible all the time, until $|H \cap X|$ is divisible by d. This final hyperplane h satisfies the theorem.

4 Concluding remarks

We were not able to generalize the hamburger theorem for $d \geq 3$ and $d+2$ or more measures on $\mathbb{R}^d$, even if instead of the condition that the hyperplane cuts at least $1/(d+1)$ of the total measure on each side, we require only that the partition is nontrivial.

Problem 14. *Let $d \geq 3$ and $r \geq d+2$ be integers. Let $\mu_1, \mu_2, \ldots, \mu_r$ be absolutely continuous positive finite Borel measures on $\mathbb{R}^d$ that are balanced in $\mathbb{R}^d$. Does there exists a hyperplane h such that for each open halfspace H defined by h, the total measure $\sum_{j=1}^{r} \mu_j(H)$ is positive and the measures $\mu_1, \mu_2, \ldots, \mu_r$ are balanced in H?*

It is easy to see that for a given d, it would be sufficient to prove Problem 14 for $r \leq 2d-1$. Indeed, suppose that $r \geq 2d$ and that $\mu_1, \mu_2, \ldots, \mu_r$ are measures balanced in $\mathbb{R}^d$. If $\mu_1(\mathbb{R}^d) \geq \mu_2(\mathbb{R}^d) \geq \cdots \geq \mu_r(\mathbb{R}^d)$, then after replacing μ_{r-1} and μ_r by a single measure $\mu'_{r-1} = \mu_{r-1} + \mu_r$, the resulting set of $r-1$ measures is still balanced in $\mathbb{R}^d$.

For the case of five measures on $\mathbb{R}^3$, our approach from the proof of the hamburger theorem fails for the following reason. If $\mu_i(\mathbb{R}^3) = 1/5$ for each $i \in [5]$, then the target polytope, now in $\mathbb{R}^5$, intersected with the boundary of the box B, does not contain a closed curve symmetric with respect to the center of B. If such a curve existed, we could apply a generalization of the Borsuk–Ulam theorem saying that if $f : S^k \to S^{k+l}$ and $g : S^l \to S^{k+l}$ are antipodal maps, then their images intersect [14, Exercise 3.*/116].

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